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An analysis of a nonlinear pendulum-type equation arising in smectic C liquid crystals

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Abstract. We consider a planar sample of non-chiral smectic C liquid crystal to which we impose a tilted static electric field which is augmented by a weak low-frequency alternating field. Under certain conditions it is known from the work by Stewart *et al* (Stewart I W, Carlsson T and Leslie F M 1994 *Phys. Rev.* E **49** 2130) that the resulting motion of the $\boldsymbol{c}$-director may be chaotic. This problem has been studied in detail in Stewart *et al* (Stewart I W, Carlsson T and Leslie F M 1994 *Phys. Rev.* E **49** 2130, Stewart I W, Carlsson T and Ardill R W B 1996 *Phys. Rev.* E **54** 6413) using a Melnikov analysis approach for a particular form of perturbation when the dielectric anisotropy is assumed to be positive. The addition of the oscillatory term to the field is therefore the cause of more complicated behaviour. In this paper we shall discuss the case when the dielectric anisotropy is assumed to be negative. We shall show that, by considering a linear approximation to the equation of motion, the stability of the $\boldsymbol{c}$-director cannot be guaranteed. Furthermore, we shall employ the harmonic balance technique to the nonlinear equation in order to determine approximations for the anticipated location of an 'escape' region in parameter space. The corresponding Melnikov criteria for negative dielectric anisotropy will also be found and compared with the approximate 'escape' region.

1. Introduction

Smectic C liquid crystals are layered structures consisting of elongated molecules where the long molecular axes are mathematically assumed to locally adopt one common direction in space, as is consistent with the usual continuum theory. We refer to this common direction by a unit vector labelled $\boldsymbol{n}$ called the *director*. Within the layers the molecules are not normal but instead are tilted on average at an angle θ with respect to the layer normal. Hence the director can be considered as lying on a cone tilted at an angle θ with respect to the layer normal. The tilt angle depends on the temperature which, for our purposes, we shall assume to be constant in order that we can make use of the continuum theory proposed in Leslie *et al* [3]. We shall also assume that the layers are of constant thickness. We denote the unit layer normal by $\boldsymbol{a}$ and the director by $\boldsymbol{n}$. The director therefore makes an angle θ with $\boldsymbol{a}$. We also introduce a unit vector $\boldsymbol{c}$ which is the projection of the director onto the smectic planes and is therefore perpendicular to $\boldsymbol{a}$. This can be seen diagrammatically in figure 1. We can make the following ansatz for $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{n}$:

$$\boldsymbol{a} = (0, 0, 1) \tag{1.1}$$

$$\boldsymbol{c} = (\cos\phi(z, t), \sin\phi(z, t), 0) \tag{1.2}$$

$$\boldsymbol{n} = \boldsymbol{a}\cos\theta + \boldsymbol{c}\sin\theta \tag{1.3}$$

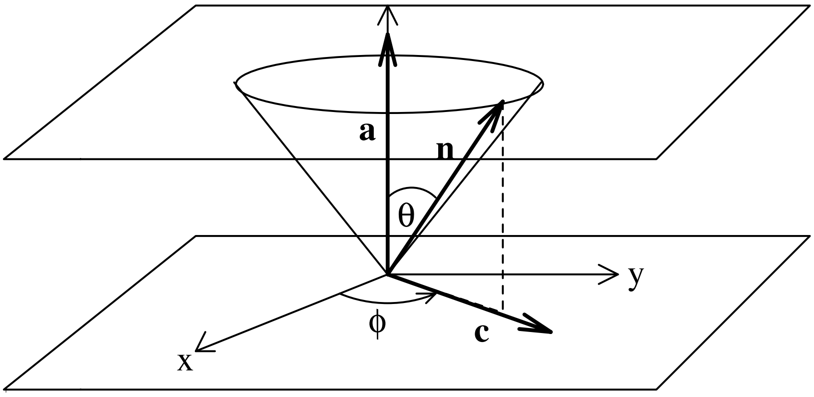

Figure 1. Mathematical set-up of a smectic C liquid crystal. The director $\boldsymbol{n}$ is at an angle θ with the layer normal $\boldsymbol{a}$. The unit vector $\boldsymbol{c}$ is the projection of $\boldsymbol{n}$ onto the smectic planes and ϕ is the phase angle of the $\boldsymbol{c}$-director.

where ϕ is the phase angle assumed to be a function of z and time t. Clearly, knowing $\boldsymbol{a}$ and $\boldsymbol{c}$ means that the orientation of $\boldsymbol{n}$ is known. The principal aim of this paper is to estimate possible predictive criteria for the onset of complex nonlinear behaviour for solutions ϕ to the governing equation (2.14), which is briefly derived below in the next section. This equation determines the behaviour of the phase angle ϕ of the $\boldsymbol{c}$-director when certain basic approximations are introduced. The possibility of such complex behaviour will be of interest to both theoreticians and experimentalists.

The techniques employed below are motivated by observations involving the stability and instability regions for the Mathieu equation (see (3.4) below). This approach requires particular constant coefficients (α_ϵ and β_ϵ) to be positive so that the methods indicated below can be used. Such positive coefficients arise naturally when the dielectric anisotropy ϵ_a is negative. In contrast to this, the techniques and results in [1,2] are applied when ϵ_a is positive. The analysis introduced in this paper allows the results from [1,2] to be extended to the case of negative dielectric anisotropy. Qualitatively, the results presented below are similar to those in the aforementioned articles. One major difference in the work presented below is that a restriction has to be made on the range of the rescaled frequency ($\bar{\omega}$) of the augmented oscillating electric field term: this is a consequence of using Mathieu's equation to initially determine the instability regions involving the frequency. Furthermore, these results create additional interest because they also happen to naturally extend known results for related nonlinear pendulum-type equations as discussed in [4–7]. A Melnikov analysis is also made here, as in [1, 2], to predict the possible parameters for the presence of transient chaotic behaviour. However, for many applications it is known that predicting parameter regions where no major stable non-rotating orbits exist is of primary importance and under these circumstances the parametrically excited pendulum is analogous to a system which allows escape from a potential well. More complex behaviour can occur for parameters within such 'escape' regions [4–7] as indicated below, including the possibility of large-time chaotic behaviour. Such results clearly have an impact upon the interpretation of the orientation of the director $\boldsymbol{n}$ in liquid crystals via the phase angle ϕ.

It should be emphasized that the results of this paper are intended to establish possible predictive criteria for the occurrence of complex behaviour in the orientation of the liquid crystal director. The actual validity of these criteria and the orientation of the director requires detailed numerical investigation. Such investigations will be possible using the results derived below for predicting the regions of the physically relevant parameters which influence the occurrence of complicated nonlinear behaviour.

2. Governing equations

We first make some remarks concerning the application of a static field before going on to discuss the case of a static field augmented with a small amplitude oscillating field. As in Stewart *et al* [1] we can an apply an external electric field to a homogeneous sample of smectic C liquid crystal at a small positive angle α to the smectic plane:

$$\boldsymbol{E} = E_0(\cos\alpha, 0, \sin\alpha) \tag{2.1}$$

where $E_0 = |\boldsymbol{E}|$ is the strength of the electric field. The dynamic equation for the $\boldsymbol{c}$-director can be written as

$$B_3\frac{\partial^2\phi}{\partial z^2} - 2\lambda_5\frac{\partial\phi}{\partial t} - \epsilon_a\epsilon_0 E_0^2(\sin\alpha\cos\theta + \cos\alpha\sin\theta\cos\phi)\cos\alpha\sin\theta\sin\phi = 0. \tag{2.2}$$

Details of the derivation of this equation can be found in [1]. The physical parameters ϵ_0 and ϵ_a are the usual (positive) permittivity of free space and the dielectric anisotropy respectively, which we assume here to be negative. Also, λ_5 is the positive rotational viscosity related to the movement of the director round the cone whose semi-vertical angle is the smectic tilt angle θ and B_3 is the positive elastic constant related to the rotation of the $\boldsymbol{c}$-director as we move from layer to layer.

We note that (2.2) can be rewritten for negative dielectric anisotropy by introducing the scalings

$$\lambda = \frac{1}{E_0}\sqrt{\frac{B_3}{-\epsilon_a\epsilon_0}} \tag{2.3}$$

$$t_0 = \frac{2\lambda_5}{-\epsilon_a\epsilon_0 E_0^2} \tag{2.4}$$

and so (2.2) becomes

$$\lambda^2\frac{\partial^2\phi}{\partial z^2} - t_0\frac{\partial\phi}{\partial t} = -(\sin\alpha\cos\theta + \cos\alpha\sin\theta\cos\phi)\cos\alpha\sin\theta\sin\phi. \tag{2.5}$$

We now give a brief account of the approximations that are made and the equation of motion which results from combining the static field with a slowly oscillating ac field. First, we assume that the tilted field angle is small such that

$$0 < \alpha \ll 1. \tag{2.6}$$

In order for complex motion of the $\boldsymbol{c}$-director to occur the static field is augmented by a weak low-frequency alternating field. We achieve the ac field by initially considering a static field and then superimposing an ac field of slowly varying frequency and so E_0 in (2.2) is replaced by

$$E_0 \to E_0[1 + (\epsilon/2)\cos(\omega\epsilon t)] \tag{2.7}$$

where $\omega\epsilon$ is the frequency of the ac field and ϵ is suitably small. We also suppose that

$$\epsilon = \xi\alpha \qquad \xi > 0 \tag{2.8}$$

which is used to make the problem more tractable. There are no known exact travelling wave solutions available for (2.5). However, the corresponding problem for positive dielectric anisotropy shown in [1] is known to possess soliton-like solutions as discussed in Schiller *et al* [8], Stewart [9] and van Saarloos *et al* [10]. Motivated by these soliton-like solutions discussed in [1] we make the following ansatz for ϕ:

$$\phi(z,t) = \phi(\tau) \tag{2.9}$$

$$\tau = \frac{z}{\lambda}\sin\theta - \frac{t}{t_0}\alpha\sin^2\theta + d \tag{2.10}$$

where d is an arbitrary constant which is then set to be

$$d = -\frac{\sin\theta}{\lambda}z_0 \tag{2.11}$$

so that when $|z - z_0| \ll 1$ is considered, further basic approximations can be made. For example:

$$\alpha\cos(\omega\xi\alpha t) \approx \alpha\cos(\bar{\omega}t) \tag{2.12}$$

where we define

$$\bar{\omega} = \frac{\omega\xi t_0}{\sin^2\theta}. \tag{2.13}$$

With these assumptions the equation of motion (2.5) can be approximated for small α and ϵ by

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} + \alpha\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = -\alpha\cot\theta\sin\phi - \frac{1}{2}[1 + \xi\alpha\cos(\bar{\omega}\tau)]\sin(2\phi). \tag{2.14}$$

This equation is of a similar form to that discussed in [1] and can be considered as an adaptation of the nonlinear pendulum-type equation considered in [4–7, 11]. Consequently, the results presented below will also be of interest to the wider scientific community. As in a similar approach in [5], it is the establishment of expected parameter ranges which may lead to complex behaviour that is the aim of this paper. Numerical work is nearly always necessary to gain accurate information about the nonlinear phenomena associated with equations such as (2.14), and this aspect of the analysis is beyond the scope of this paper. It is therefore expected that the results presented below will guide future numerical investigations not only to (2.14) but also (2.2).

3. Analysis of the linearized problem

As a preliminary study of the equation of motion of the **c**-director we shall consider the linear approximation to (2.14). This will give us some insight into the application of nonlinear techniques used later in this paper. First, we make the substitution

$$\bar{t} = \bar{\omega}\tau \tag{3.1}$$

and then we linearize in ϕ, enabling (2.14) to be rewritten as

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\bar{t}^2} + \frac{\alpha}{\bar{\omega}}\frac{\mathrm{d}\phi}{\mathrm{d}\bar{t}} + \frac{1}{\bar{\omega}^2}[1 + \alpha\cot\theta + \xi\alpha\cos\bar{t}]\phi = 0. \tag{3.2}$$

This is of the form of Mathieu's equation with damping as defined in Jordan and Smith [12, p 259]. In order that we can transform (3.2) into Mathieu's equation we now make the following transformation:

$$\phi(\bar{t}) = \exp\left(-\frac{\alpha}{2\bar{\omega}}\bar{t}\right)\rho(\bar{t}) \tag{3.3}$$

where $\rho(\bar{t})$ is simply another function in $\bar{t}$ and so (3.2) can be written as

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}\bar{t}^2} + [\alpha_\epsilon + \beta_\epsilon \cos \bar{t}]\rho = 0 \tag{3.4}$$

where

$$\alpha_\epsilon = \frac{1}{\bar{\omega}^2}\left[1 + \alpha \cot\theta - \frac{\alpha^2}{4}\right] \tag{3.5}$$

$$\beta_\epsilon = \frac{\xi\alpha}{\bar{\omega}^2}. \tag{3.6}$$

Equation (3.4) is Mathieu's equation and we can now use standard results in order to analyse the behaviour. It can be seen that stable solutions of (3.4) are clearly stable for (3.2) because of the nature of the transformation given in (3.3). Similar results for unstable solutions are more complex. Regions where instability is guaranteed can be calculated and details of these unstable regions can be found in Grimshaw [13] and Hagerdon [14]. Mathieu's equation has stable and unstable solutions depending on the values of the parameters α_ϵ and β_ϵ. Transition curves in the α_ϵ, β_ϵ-plane separating the stable and unstable solutions are easily calculated by perturbation techniques and can be found in many textbooks on nonlinear equations (e.g. [12]). We note that the following hold:

$$\bar{\omega} > 0 \qquad \alpha > 0 \qquad \xi > 0 \qquad 0 < \theta < \frac{\pi}{2} \tag{3.7}$$

and so $\alpha_\epsilon > 0$ and $\beta_\epsilon > 0$. In this region there are both stable and unstable solutions to Mathieu's equation. In order to determine which parameter values guarantee stable solutions to (3.2) via (3.4) we need to study the areas round the points $\alpha_\epsilon = \frac{1}{4}, 1, \frac{9}{4} \ldots$. If $\bar{\omega} < 1$ and $\alpha \ll 1$ then $\alpha_{\epsilon_-} > 1$ and, in general, solutions will be stable except for specified values of $\bar{\omega}$ corresponding to $\alpha_{\epsilon_-} = \frac{n^2}{4}$ where n is a positive integer greater than 2. The most likely parameter values to result in unstable solutions occur around $\alpha_{\epsilon_-} = \frac{1}{4}$. By considering (3.5) this occurs, for small α, near $\bar{\omega} = 2$. Solutions to (3.4) which are unstable may not necessarily guarantee unstable solutions to (3.2) due to the nature of the transformation (3.3). Nevertheless, stability also cannot be guaranteed and this motivates the choice of parameter values for study in the next section, in particular for $\bar{\omega} \approx 2$. Also, at this point, we make note that, for the case where the dielectric anisotropy is positive, it is well known that the solutions to (3.4) with the appropriate changes made to (3.5) and (3.6) are generally unstable for the parameter ranges shown above.

4. Melnikov criterion and harmonic balance technique

We are now in a position to consider some nonlinear techniques in an attempt to establish some results for predicting the parameter zone in which no major stable non-rotating orbits exist, termed the *escape zone* in [5]. First, we can find the Melnikov function, $M(\tau_0)$, in a manner similar to that employed in [1]. It can be shown that, for negative dielectric anisotropy, $M_+(\tau_0)$ can be written as

$$M_+(\tau_0) = \frac{\xi}{2}\pi\bar{\omega}^2 \operatorname{csch}\left(\frac{\bar{\omega}\pi}{2}\right)\left[\sin(\bar{\omega}\tau_0) - \frac{4}{\xi\pi\bar{\omega}^2}\sinh\left(\frac{\bar{\omega}\pi}{2}\right)\right] \tag{4.1}$$

and this has simple zeros if

$$\left|\frac{4}{\pi\bar{\omega}^2}\sinh\left(\frac{\bar{\omega}\pi}{2}\right)\right| < \xi. \tag{4.2}$$

The case for $M_-(\tau_0)$ is similar and it can be shown that $M_-(\tau_0) = M_+(\tau_0)$. Details of this calculation can be found in the appendix based upon the work in [1]. The method of Melnikov is known to give the region in parameter space where a homoclinic orbit or a heteroclinic orbit between two equilibrium points begins to break up. The boundary of this region, known as the Melnikov curve, is given by an equality in (4.2). The onset of chaos, in the sense of Smale horseshoes, may follow the break up of the homoclinic or heteroclinic orbits. However, it is not a necessary condition that fully chaotic motion occurs and hence parameter values lying within the region corresponding to the Melnikov analysis may or may not result in fully chaotic motion of the $\boldsymbol{c}$-director. Nevertheless, the Melnikov curve is often a good criterion for determining transient chaos [7]. It has been shown in [6] that a pitchfork and symmetry-breaking bifurcation may form a bound for the escape zone. These bifurcations can be approximated mathematically and are known in some explicit cases to give more exact bounds for the escape zone where large-time chaotic behaviour can occur [5].

Differential equations involving sinusoidal nonlinearities and forcing terms have been successfully studied using a harmonic balance criterion in [4] and [5] in which the dynamic equation arising from the parametrically excited pendulum is considered. In [11] the authors use the same technique to study a natural extension of the pendulum equation which is known to occur in the smectic liquid crystal literature. The dynamic equation in [11] arises from a study of the same physical problem outlined in this work but for a different form of perturbation. We follow the work of [4, 5] and [11] in which a study is made of parametrically excited pendulum-type equations. In particular these authors conjecture that the chaotic region can be approximated by seeking out bifurcations which occur prior to chaos. They use the method of harmonic balance to predict where these bifurcations occur. We assume that the motion of the director is sinusoidal and we study the motion which originates close to the first Mathieu zone, the region where the transition curves first intersect on the α-axis. Motivated by the initial work on the linearized problem in section 2 we therefore consider the region around the point $\alpha_\epsilon = \frac{1}{4}$ and so we choose to study only the case when the diamagnetic anisotropy is negative and $\bar{\omega} \approx 2$.

We first assume, following the method in [4] and [5], that the solution to (2.14) is of the form

$$\phi(\bar{t}) = \phi_0 + A\cos(\upsilon(\bar{\omega}\bar{t} + \beta)) \tag{4.3}$$

where $\upsilon = \frac{1}{2}$ corresponds to the primary unstable zone around the point $\bar{\omega} = 2$. In order to make further calculations clearer we also set

$$T = \tfrac{1}{2}(\bar{\omega}\bar{t} + \beta). \tag{4.4}$$

We substitute (4.3) into (2.14) to obtain

$$\begin{aligned} -\frac{\bar{\omega}^2}{4}A\cos(T) - \frac{\alpha\bar{\omega}}{2}A\sin(T) + \alpha\cot\theta\sin(\phi_0 + A\cos(T)) \\ +\tfrac{1}{2}[1 + \xi\alpha\cos(2T - \beta)]\sin(2\phi_0 + 2A\cos(T)) = 0. \end{aligned} \tag{4.5}$$

In (4.5) we make use of the identities in Abramowitz and Stegun [15, p 361]:

$$\cos(x\cos y) = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)\cos(2ny) \tag{4.6}$$

$$\sin(x\cos y) = 2\sum_{n=1}^{\infty}(-1)^{n-1} J_{2n-1}(x)\cos((2n-1)y) \tag{4.7}$$

where J_n is the Bessel function of the first kind of order n. When we equate terms which are constant or involve $\sin(T)$ or $\cos(T)$ to zero, ignoring any higher harmonics in T, equation (4.5) yields

$$\sin\phi_0\{\alpha\cot\theta J_0(A) + \cos\phi_0[J_0(2A) - \xi\alpha J_2(2A)\cos\beta]\} = 0 \tag{4.8}$$

$$-\frac{\alpha\bar{\omega}}{2}A + \frac{1}{2}\xi\alpha\cos 2\phi_0\sin\beta(J_1(2A) + J_3(2A)) = 0 \tag{4.9}$$

$$-\frac{\bar{\omega}^2}{4}A + \frac{1}{2}\xi\alpha\cos 2\phi_0\cos\beta(J_1(2A) - J_3(2A)) \\ +2\alpha\cot\theta\cos\phi_0 J_1(A) + \cos 2\phi_0 J_1(2A) = 0. \tag{4.10}$$

Equation (4.8) can be split into the symmetric solution

$$\sin\phi_0 = 0 \tag{4.11}$$

and the asymmetric solution

$$\alpha\cot\theta J_0(A) + \cos\phi_0[J_0(2A) - \xi\alpha J_2(2A)\cos\beta] = 0. \tag{4.12}$$

For the symmetric solution $\phi_0 = 0$, (4.9) and (4.10) can be suitably squared to eliminate β yielding the relevant symmetric condition

$$\frac{1}{4}\xi^2\alpha^2(J_1(2A) + J_3(2A))^2(J_1(2A) - J_3(2A))^2 \\ -\left(J_1(2A) - \frac{\bar{\omega}^2}{4}A + 2\alpha\cot\theta J_1(A)\right)^2(J_1(2A) + J_3(2A))^2 \\ -\left(\frac{\alpha\bar{\omega}}{2}\right)^2 A^2(J_1(2A) - J_3(2A))^2 = 0. \tag{4.13}$$

For light damping, $\beta \approx 0$, the asymmetric equation reduces to

$$\epsilon \equiv \xi\alpha = \frac{\alpha\cot\theta J_0(A) + J_0(2A)}{J_2(2A)}. \tag{4.14}$$

As in [5] and [11] we can solve simultaneously the condition (4.13) for the symmetric solution to occur and the condition (4.14) for the asymmetric solution to occur numerically. For a fixed $\bar{\omega}$ we substitute (4.14) into (4.13) and solve (4.13) numerically for A using a nonlinear solver noting, by the form of (4.3), that we wish to consider solutions of small amplitude, namely $A \approx 1$. We can then substitute this value of A into (4.14) to obtain the corresponding value of ϵ. By varying the values of $\bar{\omega}$ this provides a locus of points in the $\bar{\omega}, \epsilon$-plane corresponding to the symmetry-breaking bifurcation curve for fixed values of θ, α and ξ.

By considering (4.13) it is possible to determine saddle node bifurcation points as has been accomplished in [4] and [5]. These occur when the number of solutions varies as the parameter values are changed, and can be found using a vertical tangency condition where $\mathrm{d}A/\mathrm{d}\omega = \infty$. As in [11], we can define the left-hand side of equation (4.13) as f and then $\mathrm{d}A/\mathrm{d}\omega = -f_\omega/f_A$ where the subscripts denote partial differentiation. We can expand $\mathrm{d}A/\mathrm{d}\omega$ in A to find that

$$\frac{\mathrm{d}A}{\mathrm{d}\omega} = -\frac{(1 - \frac{1}{4}\omega^2 + \alpha\cot(\theta))\omega - \frac{1}{2}\alpha^2\omega}{\epsilon^2 - 4(1 - \frac{1}{4}\omega^2 + \alpha\cot(\theta))^2 - \alpha^2\omega^2}A + \mathrm{O}(A^3) \tag{4.15}$$

and so, for small A, $\mathrm{d}A/\mathrm{d}\omega = \infty$ when

$$\frac{\epsilon^2}{4} - \left(1 - \frac{\bar{\omega}^2}{4} + \alpha\cot(\theta)\right)^2 - \frac{\alpha^2\bar{\omega}^2}{4} = 0. \tag{4.16}$$

This curve in the $\bar{\omega}, \epsilon$-plane corresponds to a subcritical bifurcation and can be found by linearizing equation (2.14) in ϕ to Mathieu's equation, as in [12]. The curves corresponding to the symmetry-breaking and subcritical bifurcations are plotted in figure 2 for $\alpha = 1^\circ$

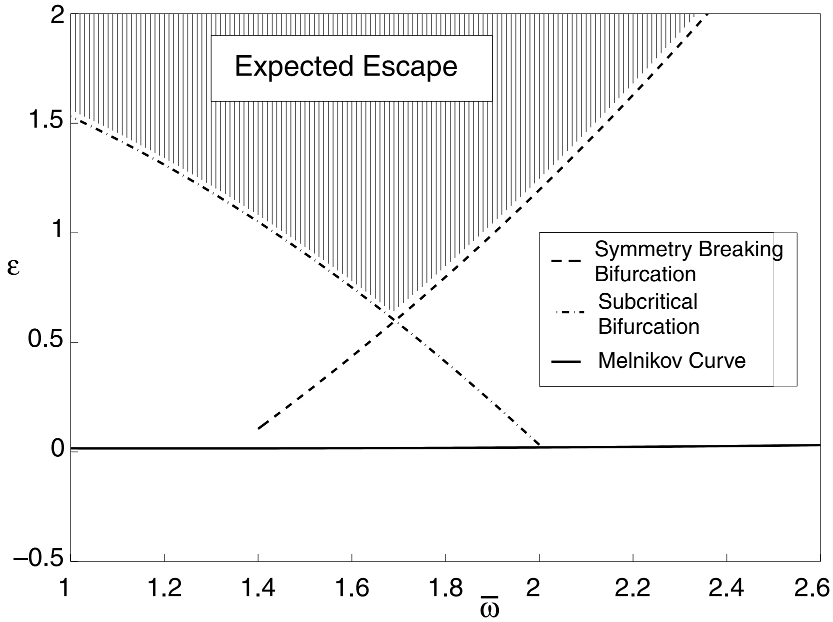

Figure 2. The symmetry-breaking bifurcation curve and the subcritical bifurcation curve plotted in the $\bar{\omega}$, ϵ-plane. Also plotted is the Melnikov curve. As expected, the anticipated escape region lies above the Melnikov curve. ϵ is related to the amplitude and slowly varying frequency effect of the superimposed oscillating electric field introduced in equation (2.7).

and $\theta = 20°$. We have also plotted the Melnikov curve and it can be seen that the escape region bounded by the two bifurcation curves lies above the Melnikov curve. This is to be expected, especially since figure 2 is reminiscent of figures 12 and 13 in [7] where the Melnikov curve generally lies below both the period-doubling and subcritical bifurcations. However, we have used the symmetry-breaking bifurcation to approximate the cascade of period-doubling bifurcations which is hard to predict analytically. The first period-doubling bifurcation is equivalent to the symmetry-breaking bifurcation for the case of symmetric systems and is often sufficiently close to the final bifurcation to provide a reasonable estimate.

Figure 2 reveals qualitatively that as ϵ increases the opportunity for complex behaviour in the solution ϕ of equation (2.14) also increases. The amplitude and slowly varying frequency of the oscillating field contribution in (2.7) to the liquid crystal problem are controlled by the magnitude of ϵ and, therefore, whenever the assumptions introduced above are feasible, the general orientation angle ϕ of the $\boldsymbol{c}$-director may exhibit complex behaviour as ϵ increases, perhaps especially in the low-frequency regime.

5. Discussion

An analysis for a special time-dependent perturbation to the dynamic equation (2.2) (leading to equation (2.14)) arising in the smectic C liquid crystal literature has revealed the possibility of predicting problem-related parameter regions where complex behaviour may be present, motivated by similar methods presented in [5,7]. The techniques employed in [5] for nonlinear pendulum-type equations have been extended to equation (2.14) in order to derive predictive criteria for the onset of complex phenomena. The main results are presented in figure 2 where an expected 'escape' phenomena region in the $\bar{\omega}, \epsilon$-plane is displayed. The parameter $\bar{\omega}$ is a rescaled frequency and ϵ is the parameter which, as introduced in equation (2.7), is a measure of the amplitude of the superimposed oscillating electric field and its slowly varying frequency. It is easily seen from figure 2 that as ϵ increases, that is, as the magnitude of the oscillating field contribution increases, the possibility of complex behaviour also increases. This is similar to what can happen for various types of nonlinear oscillations subjected to a periodic driving force such as $F\cos(\omega t)$: complex behaviour patterns can emerge as F increases leading to chaos/escape regions in the analogous ω, F-plane (see, e.g., figure 10 in [7]).

The possible escape parameter region presented in figure 2 is, of course, related directly to equation (2.14) and may act as a guide to future numerical investigations of (2.14). Such work will clearly be of wider interest because of the close relationship of (2.14) to nonlinear pendulum-type equations. The results clearly have an impact upon the liquid crystal problem outlined in section 1 when equation (2.2) is subjected to a perturbation imposed upon its electric field contribution. Section 3 reveals that the basic approximations and assumptions used in section 1 to yield the governing perturbed equation (2.14) (in this sense) may lead to complex behaviour of the solution ϕ, which is the phase angle orientation of the $\boldsymbol{c}$-director within the planes of the smectic C liquid crystal sample. It is therefore anticipated that complex/chaotic motion of the director may be present for the problem described in section 1 whenever the given approximations introduced may be appropriate to yield equation (2.14). Numerical investigation of both (2.2) and (2.14), guided by the results of section 3 above, will therefore be of interest, not only to the liquid crystal community, but also to researchers working on nonlinear dynamics related to pendulum-type equations.

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Appendix

In order to make use of Melnikov's method we need to rewrite (2.14), using (2.8), in the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix}\phi\\ v\end{bmatrix} &= \begin{bmatrix} v \\ -\frac{1}{2}\sin(2\phi)\end{bmatrix} + \alpha\begin{bmatrix} 0 \\ -v - \cot\theta\sin\phi - (\xi/2)\cos(\bar{\omega}\tau)\sin(2\phi)\end{bmatrix}\\ &= \begin{bmatrix} f_1(\phi, v)\\ f_2(\phi, v)\end{bmatrix} + \alpha\begin{bmatrix} g_1(\phi, v, \tau)\\ g_2(\phi, v, \tau)\end{bmatrix}\\ &= \boldsymbol{f}(\phi, v) + \alpha\boldsymbol{g}(\phi, v, \tau)\end{aligned} \tag{A.1}$$

where $v = \mathrm{d}\phi/\mathrm{d}\tau$.

We now consider the phase portrait of the unperturbed system. We have the same equilibrium points as with the case when $\epsilon_a > 0$ but in the current case the saddles occur when $v = 0$, $\phi = (2n+1)\pi/2$ and the centres occur when $v = 0$, $\phi = n\pi$, where n is an integer. Thus, instead of considering the heteroclinic orbit which occurs in the phase plane between 0 and π, as in [1], we instead study the heteroclinic orbit in the phase portrait for ϕ between $-\pi/2$ and $\pi/2$. The unperturbed system has a Hamiltonian, H, given by

$$H(\phi, v) = \tfrac{1}{2}v^2 - \tfrac{1}{4}\cos(2\phi) \tag{A.2}$$

from which the equilibrium points can be found and their nature determined. The upper and lower heteroclines forming the heteroclinic orbit are also the separatrices in the phase plane and occur when $H = \frac{1}{4}$ when

$$v_+ = \cos\phi \tag{A.3}$$

$$v_- = -\cos\phi \tag{A.4}$$

respectively. Since the phase plane for the case when $\epsilon_a < 0$ is the same as the phase plane for the case when $\epsilon_a > 0$, except for a phase shift of $\frac{\pi}{2}$, it is not surprising to find that the heteroclinic orbits can be found to be

$$\boldsymbol{q}_+^0(\tau) = (\phi_+^0, v_+^0) = \left(2\tan^{-1}[\exp(\tau)] - \frac{\pi}{2}, \operatorname{sech}\tau\right) \tag{A.5}$$

$$\boldsymbol{q}_-^0(\tau) = (\phi_-^0, v_-^0) = \left(\frac{\pi}{2} - 2\tan^{-1}[\exp(\tau)], -\operatorname{sech}\tau\right) \tag{A.6}$$

where, again, $-\infty < \tau < \infty$. Nevertheless the application of the Melnikov technique leads to qualitatively different results in this case, since the evaluation of the integrals for M_+ change dramatically compared to the M_+ calculations in [1].

We are now in the position to evaluate the Melnikov function but first we note that

$$\sin\phi_+^0(\tau) = \tanh\tau \tag{A.7}$$

$$\sin\phi_-^0(\tau) = -\tanh\tau \tag{A.8}$$

$$\sin 2\phi_+^0(\tau) = 2\operatorname{sech}\tau\tanh\tau \tag{A.9}$$

$$\sin 2\phi_-^0(\tau) = -2\operatorname{sech}\tau\tanh\tau \tag{A.10}$$

and so, by (A.1), from the usual definition of M_+,

$$\begin{aligned} M_+(\tau_0) &= \int_{-\infty}^{\infty} \boldsymbol{f}(\boldsymbol{q}_+^0(\tau)) \wedge \boldsymbol{g}(\boldsymbol{q}_+^0(\tau), \tau + \tau_0)\,\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} v_+^0(\tau)\left[-v_+^0(\tau) - \cot\theta\sin\phi_+^0(\tau) - \frac{\xi}{2}\cos(\bar{\omega}(\tau+\tau_0))\sin(2\phi_+^0(\tau))\right]\mathrm{d}\tau. \end{aligned} \tag{A.11}$$

Making use of (A.7) and (A.8) we obtain

$$\begin{aligned} M_+(\tau_0) &= \int_{-\infty}^{\infty} \operatorname{sech}\tau[-\operatorname{sech}\tau - \cot\theta\tanh\tau - \xi\cos(\bar{\omega}(\tau+\tau_0))\operatorname{sech}\tau\tanh\tau]\,\mathrm{d}\tau \\ &= -\int_{-\infty}^{\infty}[\cot\theta\operatorname{sech}\tau\tanh\tau + \operatorname{sech}^2\tau]\,\mathrm{d}\tau \\ &\quad -\xi\int_{-\infty}^{\infty}[\cos(\bar{\omega}(\tau+\tau_0))\operatorname{sech}^2\tau\tanh\tau]\,\mathrm{d}\tau = -2 - \xi I \end{aligned} \tag{A.12}$$

where I is the integral on the third line. This integral has been solved in [1] by integrating by parts and using Gradshteyn and Ryzhik [16, p 505] to give

$$I = -\pi\frac{\bar{\omega}^2}{2}\sin(\bar{\omega}\tau_0)\operatorname{csch}(\bar{\omega}\pi/2) \tag{A.13}$$

and hence

$$M_+(\tau_0) = \frac{\xi}{2}\pi\bar{\omega}^2\operatorname{csch}\left(\frac{\bar{\omega}\pi}{2}\right)\left[\sin(\bar{\omega}\tau_0) - \frac{4}{\xi\pi\bar{\omega}^2}\sinh\left(\frac{\bar{\omega}\pi}{2}\right)\right]. \tag{A.14}$$

Thus we will have simple zeros if

$$\left|\frac{4}{\pi\bar{\omega}^2}\sinh\left(\frac{\bar{\omega}\pi}{2}\right)\right| < \xi. \tag{A.15}$$

It is interesting to note that when $\epsilon_a < 0$ then the condition for Smale horseshoes to exist is θ independent, whereas when $\epsilon_a > 0$ the same condition is dependent on the smectic tilt angle θ.

A similar analysis for the $M_-(\tau_0)$ case can made using (A.8) and (A.10). It can be shown that, in this case, $M_-(\tau_0) = M_+(\tau_0)$ and so the condition for simple zeros to exist is exactly (A.15). (Remark: this is in contrast to the results in [1] where $M_+ \neq M_-$ in general for $\epsilon_a > 0$.)

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